Conceptual Language for Solving Equations

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Introduction/ Statement of Inquiry:

Development from procedural understanding to conceptual understanding deepens mathematical reasoning. This allows students the ability to not only regurgitate methods for solving problems, but apply these methods to other mathematical problems. Conceptual knowledge “cannot be learned by rote. It must be learned by thoughtful, reflective learning. It is a complicated web of knowledge, a network in which the linking relationships are as prominent as the discrete bits of information” (Wikipedia). Determining what students understand on a conceptual level is or should be the goal of each teacher.

To look at how conceptual understanding can be recognized within the classroom I asked: What type of language do students use during the study of solving equations that leads to their conceptual understanding of the topic? Solving equations is an integral part of teaching Pre-Algebra to seventh and eighth graders. This is the foundation for continued study into Algebra and high school mathematics. In my few years of teaching I have noticed that this topic can be a road block for understanding and reasoning. Many times the steps can be memorized, but cannot be applied outside of this originally given context. I believe students must be able to use common classroom terminology relating to solving equations while they logically explain their reasoning in order to show conceptual understanding.

The problem I find is that students do not have a strong ability to reasoning why they are completing certain steps in order to solve the equation. Responses such as “because you are supposed to” are used as explanations. These comments do not actually explain anything. What language helps explain and how can students use this language to aid in their mathematical reasoning ability? Can using very specific terminology in the classroom teaching of the topic help lead them to conceptual understanding? Requiring students to use more appropriate language relating to the topic of solving equations could help demonstrate understanding as well.

Review of Literature:

Much of the literature I read focused on mathematical discourse leading to conceptual understanding. Types of questions the teacher asks and the ability of students to formulate their own questions aid in mathematical discovery. Piccolo, Harbaugh, Carter, and Capraro examined classroom questioning in a study about the *Quality of Instruction*. Questions develop understanding and create a classroom environment evident of this. "Persistent questions by both teacher and student can help facilitate the development of mathematical understanding in students" (5). The studies these professors were looking at dove into the idea that students have questions, but are found to not be asking them. This is detrimental to the learning environment because teachers focus their instruction on student questions and responses. Questioning can take forms allowing teacher to receive different information about what the students know. Guiding questions help the students' reason with their thinking as they are initially beginning. Probing and open-ended questions, allow students the opportunity to explore mathematical topics and explain their ideas for solving the problems. Without these differences students are not required to extend their thinking.

This research about questioning can be applied to language with students' ability to speak in a way that is understandable to all learners, teachers, and others. When looking at mathematics language "the desired outcome is communicating to learn mathematics rather than learning to communicate mathematically." (Piccolo, 4) The more the class is able to discuss mathematical concepts the more reasoning and understanding is involved. Language is an important component of successful classroom discussions so that all students can be engaged. Common mathematical language must be taught and continued to develop over time.

Language is developed through the introduction of topics. Students are only able to develop this language when the context and terminology is taught. Lampert and Blunk joined in editing a book researching mathematical communication and how it impacts learning. One chapter from this book by Mary Catherine O'Conner is entitled *Language Socialization in the Mathematics Classroom: Discourse Practices and Mathematical Thinking* discusses the importance of these practices in this way; "by being involved in discourse practices that require one to challenge, to ask, to check, one is gradually socialized into adopting these as habits of mind" (26). With continued classroom discussions involving important terminology the language is ingrained in the students mind. Not only do these terms need to be used frequently; the application and reasoning behind them allow students the ability to use them with reasonability.

There is also discussion going on that higher achievement is seen when students elaborate on their explanations and make clear statements to their classmates. A hypothesis about this topic is a difficult one to research and prove because students have different performance abilities. These performance abilities can impact the use of language and confidence in using that language. The research is showing that it cannot hurt to continue expanding the use of language within the mathematics classroom to allow students the opportunity to reach higher levels of understanding and conceptual knowledge.

Continuing with the idea of mathematical discourse and discussions in the classroom, there is an application from experiences and talk at home into the classroom environment. However, many times the students to do not realize this overlap or are not willing to discuss mathematics in this form. "The mathematical experiences of arguing, or of making a claim, or providing justification, or co-constructing a definition are abstract and ungraspable except as they take place within an activity or an event. That event must involve the learner." (O'Conner 27). This means that teachers need to relate the mathematical concepts to a situation in which the students already have experiences. The example used by Mary Catherine O'Conner was a freshman discussing counterexamples. She was not able to investigate fully a counter example for x^2 always being larger than x. She could give great counterexamples for the statement that every person that is admitted into the hospital is there because of a bad diet. The concept is the same, but the hospital situation made sense where the variable situation was extremely abstract for the student to work with. There is always the possibility that the student already has hang ups over certain mathematical numbers such as fractions or decimals so they avoid them at all costs. "The research suggests that increased ability to traverse the mathematical universe would involve both more robust mathematical knowledge and scaffold practice carrying out everyday discourse routines while operating within the mathematical realm." (O'Conner 36). Students need to be given opportunities to use the discourse knowledge they already posses within mathematical contexts. The scaffolding practice will allow the students to feel comfortable using the mathematical language and provide justification for the thoughts, ideas, and steps required to solve mathematical problems. Students have the ability to reason through these situations and use these mathematical concepts as long as they are given situations where the ideas are applicable to what they know. After that applying it to more abstract concepts would be more successful for students.

Teacher questions are also an important factor of mathematical discourse within the classroom setting. Boaler and Brodie analyze in their article *The Importance, Nature and Impact of Teacher Questions* place teacher questions into categories and percentages about how often each type was used. This was done in a number of teacher classrooms using a variety of teaching methods. Teacher questions required about 30% of the classroom time in reformed classrooms and the traditional classrooms were less than 20%. The category that I found to be most interesting when classifying question types as "Learning Terminology." The description of this type of question is: "once ideas are under discussion, enables correct mathematical language to be used to talk about them." (776). The teacher has to incorporate this form of questioning into the classroom setting or the students will not understand the type of terminology that they are expected to use with these mathematical concepts. It seems that they use this idea of terminology as both symbolic mathematical language and spoken mathematical language. Students must feel comfortable using both of these to have the ability to discuss and justify mathematical work.

Modes of Inquiry:

My research was completed in a Christian private school located in Miami, Florida. Both seventh and eighth grade classes of Pre-Algebra were used. It is difficult to determine exactly how much conceptual understanding is retained so individual and group activities were used. Research was completed towards the end of this topic of solving equations to receive the most accurate version of language that the students would be using. It was important to determine different forms in which students learn and also the different forms in which students are able to express their thoughts. Verbal, visual, and kinesetic learners are the most recognizable in the classroom setting. These learning styles may impact students' ability to explain the topic. Words, models, drawings, and actions may all be used to reason through mathematics study. Because of these differences I wanted to give a few options for the students to then investigate the type of language that they used in their explanations.

First, I used a very informal mode of data collection. The students were assigned a page and select problem numbers for homework out of the textbook. This came during the study of our second chapter involving solving equations, three weeks into the topic of equations. Students were not completely proficient in solving equations at this point, however they had a good idea of the steps. The problems assigned asked the students to explain the three different types of solutions that are possible when solving equations. “Explain how to determine if an equation has no solution, one solution, or if all numbers are solutions. Use examples with your explanation.” (Glencoe & Mcgraw-Hill, 252) Students were not aware that I was going to be looking specifically at this question when I graded the homework assignment. I choose to not inform the students in order to determine how much importance they placed on explaining their thinking and reasoning behind their mathematical work.

The second activity required the students to complete a reasoning worksheet. My classroom is set up with daily review at the beginning of the class period; this activity was out daily review one day. I gave the students two equations; one on each sides of the piece of paper, set up with a table to complete containing mathematical symbols and reasoning. This was the first time we had set up our reasoning in this format. Usually the students are just talking through their reasoning or writing a brief sentence to explain their answer in a homework problem. The goal of this assignment was to see what the students were able to write down as an explanation as opposed to speaking and discussing as we usually do with in the classroom. This would give me an opportunity to look at the specific terminology the students wrote down to explain their steps for solving.

Lastly, students were put into groups for peer teaching activity. This peer teaching was also used as a way of reviewing some of the previously learned material. Students were put in partners to look at a type of equation that they would be teaching to another pair of students. They were given two example equations that they need to use within their teaching. The other pair of students pretended as if they were just learning about this mathematical topic for the first time. Clear explanations and understandable language was expected. Out of each of my class periods I decided to video record a group working together on this peer teaching assignment. This way I would be able to go back and record the types of language and explanations the students used. Without video recording it would have been very difficult to remember all of the explanations that were used so the recording gives me an opportunity to go back and review the language that students use when talking with each other.

I was originally thinking about having the students work during the process of a few days with algebra tiles. They would have worked to solve a number of equations and done this in small groups similar to those of the peer teaching activity. I decided that this method of research was not going to be extremely plausible because the students were not accustomed to using algebra tiles. Since I did not introduce algebra tiles when we began our study of solving equations, I felt that It would be more detrimental to the students to go back and try to learn with algebra tiles after having already learned the procedure and are working to a very conceptual understanding of the topic. The time it would have required to feel comfortable learning how to use algebra tiles would have taken away

Results:

Data collection number on coming from students' homework assignment showed a lack of importance placed on reasoning by the students. I graded the full assignment and then kept it longer to analyze their answers to the reasoning question.

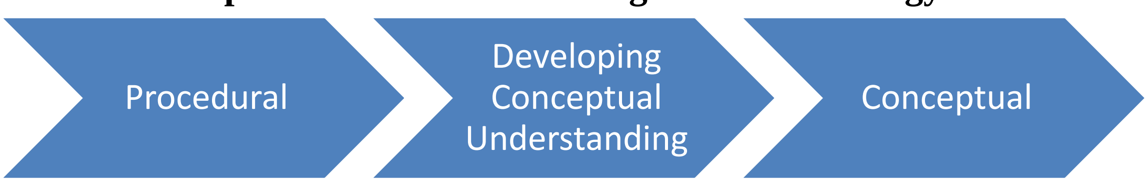
Question: Explain how to determine if an equation has no solution, one solution, or if all numbers are solutions. Use examples with your explain.

Answer (textbook given): If you get an answer like 5 = 5 which is always true, all numbers are solutions. If you get an answer like 5 = 9 which is never true, there are no solutions. If you get an answer like x = 8, there is one solution.

Total # of Students	Total # of Homework Assignments Turned in.	Total # of Answers to this Question.	Total # of Answers Copied Straight from Textbook	Total # of Answers Including only Examples	Total # of Answers Including only Reasoning	Total # of Answers Including both Reasoning and Examples
83	70	41	2	17	18	6

During the process of analyzing I began to organize student responses on a spectrum of procedural to conceptual understanding. I included along the spectrum classroom terminology and reoccurring terminology I heard and saw in student responses. With using a spectrum responses fall in many different areas and terminology used can include both procedural and conceptual understanding.

Spectrum of Understanding with Terminology



Procedural	Developing Conceptual Understanding	Conceptual
Subtract/ Add	Each Side/Same Side	Distributive Property
Multiply/Divide	Cancel Out/Get Rid Of	Inverse Operations
Solve	Eliminate	Isolate the Variable
Make Sides Easy	Opposite Operations	Combine like Terms
Simplify		

The data for this question was difficult to fit into the spectrum because it did not require students to explain step by step their reasoning. Types of solutions they were reasoning out. They needed to use the idea of "like terms" in their reasoning to show good conceptual understanding with what happens to the variables. In this case the student worked with this idea:

> No solution, same number before each variable ex: 8c + 1 = 8c + 2
> One solution, different number before variable ex: 12y + 5(y – 6) = 4
> All numbers, the equation is absolutely the same ex: 10c + 1 = 10c + 1

Conceptual understanding is shown first by the student including mathematical examples to support his written explanation. Even without exact classroom terminology ideas such as coefficients and like terms are used with his discussion about "variables". This student responded with the most advanced conceptual understanding. In contrast, the 18 students that responded with a response falling in the developing conceptual understanding word looked like this:

> No solution: if there is no variable and they do not equal. 6 = 9
> One solution: if a variable equals a number. x = 2
> All numbers: if both sides are equal. 4 = 4

The only additional reasoning that is shown in these responses is the students' ability to match understanding of the final step in the equation problem to written words. Students did not seem to understand that the question was asking for reasoning about how the sides of the equation came to be in the end not just what it reads. The other responses were completely procedural due to the fact that they only wrote down examples for the solutions. The reasoning portion of the questions was skipped.

The difficulty that I had using this question as a solid form of data is that not all students turned in their homework assignment or a great number of those that did turn in the assignment skipped the question I was using to analyze. I could not get a complete overview of all my students. Also, with only being a homework assignment many students do not do their best mathematical work. In the middle school students' mind homework is not the most important part of their mathematics course grade. One problem will not drop their grade that much, so if it is not fully answered or not answered at all it will not make a difference. This data shows me that students need to be required to write or produce in some form more mathematical explanations and reasoning so that they are comfortable completing these tasks.

The second data collection activity required the students to fully reason out their thoughts in a side by side mathematic symbols and reasoning table. Students were more focused on producing good work than the first data collection. From being a class assignment and worksheet that had to be turned in. Most students filled it out completely leaving no reasoning boxes blank. I had 80 students in class this day who responded to two equations. I used the same spectrum to organize the student responses to the equations:

1.) $2x - 8 = 10$ 2.) $7w - 6 = 3(w + 6)$

Of those 80, all but 2 of the responses solved the problem correctly on the mathematical symbol column. With these results it is evident that the students learned the appropriate procedure to successfully solve an equation. Appropriate mathematical I analyzed the reasoning side of the table for the terminology listed on my spectrum. I counted the total number of types of terminology used to explain different parts of the equation for each student. The table shows the number of responses for total amount of terminology for each given equation.

Equation	**0 terms**	**1 term**	**2 terms**	**3 terms**	**4 terms**	**5 terms**
1	15	35	23	6	0	NA
2	7	25	26	14	5	1

Equation 1 requires fewer steps for solving than the second equation. Four types of terminology would be the most that a student is capable of applying to the equation. The majority of the responses fell into the 1 or 2 types of terms. With the maximum being 4 types of terms this means that the majority of the student responses fall in the developing conceptual understanding category and a little bit to the lower end of that arrow on the spectrum.

The two types of terminology that were used in the 1 term category were: “to cancel out” and “isolate the variable”. Isolating the variable is a very conceptual idea and the first part of solving equations that was introduced in the classroom. The idea of “canceling out” comes from the students attempting to reason how to move parts of the equation for example:

Symbols	Reasoning Behind Step
2x – 8 = 10	Given problem
+ 8 + 8	Cancel the 8 (eliminate the 8)
2x = 18	Divide both sides by 2
x = 9	x = 9 is the solution to the problem

This is a difficult step for them to explain “why” requiring them to be in the developing conceptual understanding, middle of the spectrum.

“Canceling out” was also a used frequently with the phrase, “to both sides” or “to each side” in the 23 responses that contained 2 terms. This idea is a big part of solving equations and shows conceptual understanding of how equations require both sides to be balanced. I placed these into the developing conceptual understanding, but on the end of conceptual understanding.

For example: 2x – 8 + 8 = 10 + 8 “You add 8 to both sides because you do the opposite to cancel out and leave the variable.”

This example not only includes the common idea canceling out numbers and doing the same operations to both sides of the equation it adds in a third term; leave the variable. By reading this one sentence explanation the student understands how equations must be set up and the goal of the solution. This response falls in the conceptual understanding category even if the exact terms are not used. Many times the students are able to use their own terms and create mathematical sense for themselves. As long as it mathematically works, there is no reason to require the use of specific terminology.

Equation 2 required more steps to solve so there was the possibility of uses some form of 5 different types of terminology. If a student is able to use two forms of terminology it would place them directly in the middle of the spectrum developing conceptual understanding. The 6 students that used 4 or 5 terms I placed in the conceptual understanding part. The use of 3 terms would place the students in between the arrows of developing conceptual understanding and conceptual understanding. Use of 1 term places the students between the procedural understanding and developing conceptual understanding arrows. Most students fell just below or directly in the developing conceptual understanding similarly to the first equation. I was impressed with how many were able to use 3 or more terms. The terminology used by all responses except the 7 that did not use any specific language was "distributive property". This is a topic that has been used not only in the context of solving equations, but in simplifying expressions. These ideas students have been seeing in their mathematics courses since fifth grade at my school. They are very comfortable using the distributive property and in explain the reasoning behind the distributive property. I was not surprised to see so many responses include this idea.

Since that term was very easy to include, many of the response were moved up a category in the number of terms used. With having already completed reasoning for equation 1, the students were able to apply those same terms to the second equation. A few times, although not extremely common, students actually were more comfortable reasoning out the second equation and were able to use more terminology. This could be from the practice of writing out their reasoning with the first equation. These were two of the responses that used either 4 or 5 types of terminology and showed a strong conceptual understanding of solving equations. The red words show the terminology that was counted.

Student 1:

$7w - 6 = 3(w + 18)$	
$7w - 6 = 3w + 18$	Distribute so there are like terms.
$7w - 6 + 6 = 3w + 18 + 6$	Add 6 to each side to cancel out.
$7w - 3w = 3w - 3w + 24$	Subtract 3w to put like terms on the right side.
$4w \div 4 = 24 \div 4$	Divide both sides by 4 to isolate the variable.
$w = 6$	Check Solution:
	$7(6) - 6 = 3(6 + 6)$
	$42 - 6 = 3(12)$
	$36 = 36$

Student 2:

$7w - 6 = 3(w + 6)$	First distribute so that you can get rid of the parenthesis and solve for w.
$7w - 6 = \cancel{3w} + 18$	Trying to simplify so that it would be easier to solve and get rid of a w on one side.
$-3w \quad\quad \cancel{-3w}$	
$4w - 6 = 18$	Add + 6 and negative 6 to get rid of the -6. Add a +6 to both sides.
$+6 \quad +6$	
$\frac{\cancel{4}w}{\cancel{4}} = \frac{24}{4}$	Isolate the w by dividing 4 from both sides.
$w = 6$	The answer is the solution of $24 \div 4$ which is 6 so $w = 6$.

Both of these examples are using terminology to talk about all parts of solving the equation. The most difficult terminology for the students to use is the idea of inverse operations. The only way students were able to explain an inverse operation on these two equations was to write out the mathematical steps for doing that process. This concerns me with their understanding of what an inverse operation is and why it is important to understand how to use the idea of inverse operations. Student 2 shows a little bit more understanding through his mathematical symbols. He uses the same format that I used when working out problems with the students in class. The line through the 3w – 3w shows that these are becoming zero or what we have been calling "canceling out". This is the process of using inverse operations and shows that when opposite signs or operations are used the numbers equal zero so that the side of the equation is simpler. More emphasis needs to be placed on this portion of equation study.

The third form of data came from a peer-teaching activity. This activity was very difficult for the students, partly because we had not done much peer-teaching before and partly from having difficulty explaining their process and reasoning. The goal of this data was to see if different terminology was used when verbally explaining as opposed to written explanations, however I did not find there to be a difference. Actually with the verbal explanations students were more consistently using procedural terms. Talk was based around the steps and the mathematical language that explained these steps. "Use the distributive property and then you have to get the variable by itself so you subtract 10 and then you have to divide that by 2 to get your answer." Similarly to the written data the five groups recorded all used the phrase "get the variable alone" and "distributive property". These two ideas dealing with equations have the strongest conceptual understanding within my mathematics classroom. Students need more opportunities to explain their reasoning to classmates so they are able to be comfortable using common mathematical terminology. Many of the groups struggled to get out their ideas even when I knew they were able to write down the language that their classmates understand and goes along with solving equations.

My data shows that when looking at conceptual understanding there is a variety of factors that must be taken into consideration. Students will be at different levels of the spectrum and the placement of a student's understanding can be helpful using language, but it can also continually move. The terminology used may be common classroom terminology or it may be the students own terminology that makes the mathematics reasonable to them. Neither is right nor wrong as long as the terminology makes sense and is mathematically correct. The wording is not the important part it is the students' ability to determine why the math is reasonable and demonstrate their understanding to others.

Conclusions and Limitations:

Throughout this inquiry it has not only opened my eyes to the type of language I use topic introductions and explanations, but also to the timing and way questions are asked within the classroom. Mathematical reasoning and conceptual understanding needs to become deeper for students. It became very evident that students need to be given many opportunities to use mathematical language because the easy solution is to list the steps that they wrote in mathematical symbols. This reason cannot be accepted because it does not show their ability of understanding. Without practice and examples of how to reason mathematically students are not given the tools necessary to succeed.

One of the problems I had with the collection of data was that since students have a difficult time explaining and reasoning on their own, they ask for help. Many of the results I received were very similar to one another or used my helping words. It is very difficult to tell which terms individual students use when this occurs. If I have to help then how can I classify the response as having conceptual understanding? It is not completely their thoughts.

The readings I studied focused more on mathematical discourse and the questioning that is done within the classroom. Along with language, questions help lead to conceptual understanding. With this thought in mind I wonder how the students would react to explaining if specific questions were asked before this topic of study. If students are required to ask a certain amount of questions in relation to this topic how does this deepen their understanding? If they are able to ask probing questions about the material would this then deepen their actual conceptual understanding of the material?

Next Steps:

If I was going to follow up this inquiry looking at conceptual understanding of solving equations I would spend some time teaching my students how to reason appropriately. The second semester would be a great time to teach these strategies. When it is final exam time we could do a similar activity where students need to work through steps of a problem using mathematical symbols and then also writing out mathematical reasoning. I could compare this to my initial results and see if more terminology is evident. This would also help strengthen the reasoning behind many other topics. I think this is very important because the students need to be given opportunities to learn what is expected for reasoning.

I have also learned the importance of questioning. I ask a lot of questions but have realized that there is not a great number of them that ask for more than just a procedural understanding of the material. This leads to the fewer conceptual understanding results I collected as opposed to what I had anticipated. I have started looking more closely at the types of questions that I ask and have planned out questions for certain lessons to help lead my students to reason and deepen their thinking.

In conclusion, questioning, language, and conceptual understanding hugely impact the mathematical reasoning of students. Students must learn how to communicate with peers, teachers, administration, and others in a way that uses common language and terms. Without this ability it is difficult to see the knowledge that these students posses. Mathematics should be reasonable and students need to be given the right tools to learn and demonstrate their understanding of what they have learned mathematically.

Bibliography

Boaler, J., & Brodie, K. (2004). The Importance, Nature and Impact of Teacher Questions. *Proceedings of the Twenty-sixth annual meeting of the North American Chapter of International Group for Psychology of Mathematics Education.* 2, 773 – 782.

Glencoe & McGraw - Hill (2010). Pre-Algebra. Columbus: Glencoe.

Lampert, M., & Blunk, M. (1998). Talking Mathematics in School: Studies of Teaching and Learning. New York: Cambridge.

Piccolo, D., & Harbaugh, A., & Carter, T., & Capraro, M., & Capraro, R. (2008). Quality of Instruction: Examining Discourse in Middle School Mathematics Instruction. Journal of Advanced Academics, 19, 376 – 410.

Wales, J. Wikepedia. at http://wik.ed.uiuc.edu/index.php/Conceptual_knowledge (2006, April 28).